

Quommutator deformations of $\mathfrak{osp}(2,2)$

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Abstract

We analyse the Witten-Woronowicz's type deformations of the Lie superalgebra $\mathfrak{osp}(2,2)$ and obtain a deformation parametrized by three independent parameters. For some of these algebras, finite dimensional representations are formulated in terms of a finite difference operator, providing operators that are relevant for the classification of quasi exactly solvable, finite difference equations. Similar representations are also pointed out for the Lie superalgebra $\mathfrak{osp}(1,2)$.

1 Introduction

The study of deformations of classical Lie algebras and superalgebra has attracted much attention in the last few years. Many types of deformations have been proposed. In some of them, the (anti)commutators of the generators are set equal to non linear (in fact transcendental) functions of some of the generators, they are the Drinfeld-Jimbo deformations [1, 2]. Another type of deformation consists in keeping the right hand sides of the structure relations linear in the generators while the (anti)commutators in the left hand side are deformed into quommutators. For the deformations of the second type, also called of Witten-Woronowicz's type [3, 4], all the generators are treated on the same footing and the quommutator relations provide a set of simple rules to write the elements of the enveloping algebra in a canonical form; they define a normal order. This last property is crucial for the kind of applications that we have in mind, namely the classification of the quasi exactly solvable (QES), finite difference operators.

The QES (differential or finite difference) equations refer to a class of spectral equations for which a finite part of the spectrum can be obtained by solving a system of algebraic equations [5, 6]. The QES operators defining the QES equations are therefore intimately related to the linear operators preserving a finite dimensional vector space of smooth functions. Typically this vector space is a direct sum of spaces of polynomials of given degree.

There is a close relation between QES differential operators and the theory of algebras. Indeed, some basic QES operators correspond to the generators, in a suitable representation, of an abstract algebra (e.g. in the case of scalar QES operators of one variable, it is the Lie algebra $\mathfrak{sl}(2)$ [7]) and the generic QES operators appear as the elements of the enveloping algebra. With the aim to classify the QES operators it is therefore desirable to possess a set of normal ordering rules for the generators.

If we want to describe the algebraic structures underlying the QES finite

difference equations, the representations of some deformed algebra seem to emerge in a natural way. For example, the scalar QES finite difference operators in one variable [7] are related to the Witten deformation of $\mathfrak{sl}(2)$.

The crucial role played by the Lie superalgebra $\mathfrak{osp}(2,2)$ in the study of QES systems of two equations [8, 9] encouraged us to study the Witten-Woronowicz's type of deformations of this algebra. This is the topic of the next section.

2 Deformations of $\mathfrak{osp}(2,2)$

The Lie superalgebra $\mathfrak{osp}(2,2)$ has eight generators, four of them (the bosonic ones) assemble into a $\mathfrak{gl}(2)$ subalgebra. In order to make easy the comparison of our result with ref.[10] we note these generators $E_{11}, E_{22}, E_{12}, E_{21}$. The remaining (fermionic) generators split into two doublets under the adjoint action of the $\mathfrak{gl}(2)$ subalgebra, we denote them V_1, V_2 and \bar{V}_1, \bar{V}_2 .

For later convenience, we define the quommutator and antiquommutator respectively as

$$[A, B]_q = AB - qBA \quad , \quad \{A, B\}_q = AB + qBA \quad (1)$$

where q is the deformation parameter.

We studied the quommutators deformations of $\mathfrak{osp}(2,2)$ subject to the following requirements:

- each commutator (resp. anticommutator) defining the classical algebra is replaced by a quommutator (resp. antiquommutator) defined in (1) with its own parameter q .
- the couples of generators which (anti)commute are imposed to (anti)quommute.
- the structure of the deformed algebra relates the (anti)quommutators of any couple of generators to linear combinations of the generators but $[E_{12}, E_{21}]_q$ which is allowed to depend quadratically on the fermionic generators.

The most general deformation of $\text{osp}(2,2)$ obeying these restrictions and compatible with associativity depends on three parameters, say p, r, s . The different relations read as follows

- for the fermionic-fermionic relations

$$V_1^2 = V_2^2 = 0 \quad , \quad \overline{V}_1^2 = \overline{V}_2^2 = 0 \quad (2)$$

$$\{V_1, V_2\}_{\frac{p}{sr}} = 0 \quad , \quad \{\overline{V}_1, \overline{V}_2\}_{\frac{s}{pr}} = 0 \quad (3)$$

$$\begin{aligned} \{\overline{V}_1, V_1\} &= E_{11} \quad , \quad \{\overline{V}_2, V_2\} = E_{22} \\ \{\overline{V}_1, V_2\}_{psr} &= E_{12} \quad , \quad \{\overline{V}_2, V_2\}_{ps/r} = E_{22} \end{aligned} \quad (4)$$

The last equations can be seen as defining the bosonic operators E_{ij} .

- for the fermionic-bosonic relations

$$\begin{aligned} [E_{11}, V_1] &= 0 \quad , \quad [E_{22}, V_1]_{s^2} = V_1 \\ [E_{21}, V_1]_{ps/r} &= 0 \quad , \quad [E_{12}, V_1]_{sr/p} = -\frac{sr}{p} V_2 \\ [E_{11}, V_2]_{p^2} &= V_2 \quad , \quad [E_{22}, V_2] = 0 \\ [E_{21}, V_2]_{p/sr} &= -\frac{p}{sr} V_1 \quad , \quad [E_{12}, V_2]_{psr} = 0 \end{aligned} \quad (5)$$

$$\begin{aligned} [E_{11}, \overline{V}_1] &= 0 \quad , \quad [E_{22}, \overline{V}_1]_{1/s^2} = -\frac{1}{s^2} \overline{V}_1 \\ [E_{21}, \overline{V}_1]_{pr/s} &= \overline{V}_2 \quad , \quad [E_{12}, \overline{V}_1]_{1/psr} = 0 \\ [E_{11}, \overline{V}_2]_{1/p^2} &= -\frac{1}{p^2} \overline{V}_2 \quad , \quad [E_{22}, \overline{V}_2] = 0 \\ [E_{21}, \overline{V}_2]_{r/ps} &= 0 \quad , \quad [E_{12}, \overline{V}_2]_{s/pr} = \overline{V}_1 \end{aligned} \quad (6)$$

- and finally for the bosonic-bosonic operators

$$[E_{11}, E_{22}] = 0 \quad , \quad (7)$$

$$[E_{11}, E_{21}]_{1/p^2} = -\frac{1}{p^2}E_{21} \quad , \quad [E_{22}, E_{21}]_{s^2} = E_{21} \quad (8)$$

$$[E_{11}, E_{12}]_{p^2} = E_{12} \quad , \quad [E_{22}, E_{12}]_{1/s^2} = -\frac{1}{s^2}E_{12} \quad (9)$$

$$[E_{22}, E_{12}]_{s^2/p^2} = E_{11} - \frac{s^2}{p^2}E_{22} + (s^2 - 1)V_1\overline{V}_1 - \frac{s^2}{p^2}(p^2 - 1)V_2\overline{V}_2 \quad (10)$$

The three deformation parameters p, r, s are intrinsic and cannot be eliminated by a rescaling of the generators. This is seen easily from the way these parameters enter in the different coefficients defining the quommutators. The undeformed $\text{osp}(2,2)$ algebra is recovered in the limit $p = s = r = 1$.

Decoupling the fermionic generators from eq.(10) leads, together with (7), (8) and (9), to the two parameters deformation of $\text{gl}(2)$ obtained in ref. [10] (a four parameters deformation of $\text{gl}(2)$ was further obtained in ref.[11]). The parameter r of the deformation affects only the relations involving the fermionic generators.

3 Representations

Restricting the three parameters p, r, s in the algebra above to the case

$$p = q^{\frac{1}{2}} \quad , \quad s = q^{-\frac{1}{2}} \quad , \quad r = 1 \quad (11)$$

leads to a one parameter deformation of $\text{osp}(2,2)$ parametrized by q which we will refer to as $\text{osp}(2,2)_q$. It is equivalent, up to a rescaling of the generators, to the deformation discussed in [12]. Like for the Witten type deformation of $\text{sl}(2)$, it is possible to construct some representations of $\text{osp}(2,2)_q$ in terms of a finite difference operator D_q :

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x} \quad , \quad D_q x^n = [n]_q x^{n-1} \quad , \quad [n]_q \equiv \frac{1 - q^n}{1 - q} \quad (12)$$

Some of these representations act on the vector space $P(n-1) \oplus P(n)$ (denoting the set of couples of polynomials whose first (resp. second) component is of

degree at most $n - 1$ (resp. n) in the variable x). The fermionic generators are represented by

$$\begin{aligned} V_1 &= \sigma_-, & V_2 &= x\sigma_- \\ \overline{V}_1 &= q^{-n}(xD_q - [n]_q)\sigma_+, & \overline{V}_2 &= q^{-1}D_q\sigma_+ \end{aligned} \quad (13)$$

and the operators representing the E_{ij} 's are easily constructed through eq.(4). The enveloping algebra constructed with the four operators (13) generates all the 2×2 matrix, finite difference operators preserving $P(n-1) \oplus P(n)$. Accordingly, these operators are quasi exactly solvable. Note that more general representations of $\text{osp}(2,2)_q$ can be constructed [12]; they preserve the vector space $P(n) \oplus P(n+1) \oplus P(n-1) \oplus P(n)$ (using an obvious notation).

4 Deformations of $\text{osp}(1,2)$

Recently a Witten type deformation of the super Lie algebra $\text{osp}(1,2)$ was constructed [13]. We showed that this deformation is the only one to fulfil the three requirements of sect. 2 and we constructed the representations of it this algebra in terms of the operator (12). Again the space of the representation is $P(n-1) \oplus P(n)$ and the two fermionic generators of $\text{osp}(1,2)$ are represented by 2×2 matrix operators. Using exactly the same notation as in ref.[13] we find

$$V_- = \begin{pmatrix} 0 & D_{q^2} \\ 1 & 0 \end{pmatrix}, \quad V_+ = \begin{pmatrix} 0 & q^{-2n}(xD_{q^2} - [n]_{q^2}) \\ x & 0 \end{pmatrix} \quad (14)$$

The three bosonic operators, H, J_-, J_+ can then be computed through the structure of $\text{osp}(1,2)$, namely

$$H = \{V_-, V_+\}_q, \quad J_- = \{V_-, V_-\}_q, \quad J_+ = \{V_+, V_+\}_q \quad (15)$$

Corresponding to this $2n + 1$ dimensional representation, the Casimir operator (eq.(13) in ref. [13]) has a value $C = -\frac{1}{2}[-n - \frac{1}{2}]_{q^2}$

5 Concluding remarks

The most interesting examples of QES systems are related to the algebra $\text{osp}(2,2)$, e.g. the relativistic Coulomb problem and the stability of the sphaleron in the abelian Higgs model [9]. Moreover, the representations of $\text{osp}(2,2)$ formulated in terms of differential operators provide the building blocks for the construction of the QES operators preserving the vector space $P(m) \oplus P(n)$ for any positive integers m, n [9]. This result and the recent applications of discrete equations in quantum mechanics [14] have encouraged us to study the deformations of $\text{osp}(2,2)$ formulated in terms of quommutators. Our result can be generalized in several directions; for instance the operators (13) can be used to construct the QES discrete operators preserving the vector space $P(m) \oplus P(n)$. The algebraic structure underlying these operators is probably determined by a series of deformed non linear superalgebras indexed by $|m - n|$. Another possibility is to study the deformations of the Lie superalgebra $\text{spl}(V+1,1)$ for $V > 1$ (remembering the equivalence of $\text{osp}(2,2)$ with $\text{spl}(2,1)$). The result of ref. [15] suggests that some representations will be formulated in terms of finite difference operators depending of V variables. Finally it would be instructive to relate the two parameter deformation of $\text{osp}(2,2)$ obtained in ref.[16] to the deformation constructed here. The occurrence of some mapping between the two structures would allow to transport the Hopf structure elaborated in [16] into our deformation.

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References

- [1] V. Drinfeld, Proc.Intern.Congress of Mathematicians (Berkeley, 1986) 798.
- [2] M. Jimbo, Lett. Math. Phys. 10 (1985) 63.
- [3] E. Witten, Nucl. Phys. B **330**, 285 (1990).
- [4] S. Woronowicz, Commun. Math. Phys. **111** 613 (1987).
- [5] A.V. Turbiner, Comm. Math. Phys. **119**, 467 (1988).
- [6] A.G. Ushveridze, *Quasi exact solvability in quantum mechanics* (Institute of Physics Publishing, Bristol and Philadelphia, 1993).
- [7] A.V. Turbiner, J. Phys. A **25**, L 1087 (1992).
- [8] M.A. Shifman and A.V. Turbiner, Comm. Math. Phys. **120**, 347 (1989).
- [9] Y. Brihaye, P.Kosinski, J. Math. Phys.**35**, 3089 (1994).
- [10] D.B. Fairlie and C.K. Zachos, Phys.Lett.B **256** 43 (1991).
- [11] D.B. Fairlie, J. Nuyts, J. Math. Phys.**35**,3794 (1994).
- [12] Y. Brihaye, S.Giller, P.Kosinski, *On the relations between $osp(2,2)$ and the quasi exactly solvable systems* hep-th 9611226.
- [13] W. Chung, *On Witten-type deformation of $osp(1/2)$ algebra*, hep-q.alg 9609015.
- [14] P.B. Wiegmann, A.V. Zabrodin, Nucl. Phys. B **451**, 699 (1995).
- [15] Y.Brihaye, J.Nuyts, *The hidden symmetry algebras of a class of quasi exactly solvable multi dimensional equation*, Mons-University preprint, october 1995.
- [16] P. Parashar, J. Phys. A **27** (1994) 3803.